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SPLIT RING SPECTRA AND SECOND PERIODICITY FAMILIES IN STABLE HOMOTOPY OF SPHERES

LIN JINKUN

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§1. INTRODUCTION

IN RECENT years, many infinite families in the stable homotopy groups of spheres $\pi_*(S^o)_p$ had been found aside from the im J. In [15, 16], Smith proved that for any prime $p \geqslant 5$, there are $\beta_t (t \geqslant 1)$ and $\beta_{tp/j} (t \geqslant 1, 1 \leqslant j \leqslant p-1)$ families in $\pi_*(S^o)_p$, these are obtained by construction of some appropriate self maps of 4 cells spectrum $V(1)$. Almost at the same time, S. Oka [9] and Zahler [19] also obtained $\beta_{tp/j} \in \pi_*(S^o)_p$ $(t \geqslant 1, 1 \leqslant j \leqslant p-1)$ and [19] showed that these infinite families are detected by E_2 term $\mathrm{Ext}^{2,*}_{BP_*BP}(BP_*, BP_*)$ of the Adams–Novikov spectral sequence based on BP theory.

BP method seems to be more powerful. Miller, Ravenel and Wilson [7] gave a complete description of the generators of the above Ext^2. Their description of the generators $\beta_{t/j,i} (\beta_{t/j,i} = \beta_{t/j}$ if $i = 1)$ suggest that if there exists certain 4 cells spectrum $V_{j,i}$ such that $BP_*(V_{j,i}) = BP_*/(p^i, v_1^j)$ and a self map f of $V_{j,i}$ such that the induced BP_* homology homomorphism $f_* = v_2^t$, then the elements in $\pi_*(S^o)_p$ which are detected by $\beta_{t/j,i}$ are obtained directly, where v_i are the generators in the coefficient ring BP_*. Since [7] reveals some periodicity phenomenas, the elements in $\pi_*(S^o)_p$ detected by $\beta_{t/j,i}$ are called second periodicity elements in the literature.

Based on Toda's result on $d_{2p-1}(\beta_{p/p}) \neq 0$, Ravenel [13] proved that $d_{2p-1}(\beta_{p^n/p^n}) \neq 0$ for $n \geqslant 1$, where d_r are differential in the Adams–Novikov spectral sequence. That is to say, β_{p^n/p^n} dies in the spectral sequence and other $\beta_{tp^n/j} \in \mathrm{Ext}^2$, i.e. for $t \neq 1$ or $1 \leqslant j < p^n$, may detect elements in $\pi_*(S^o)_p$. In this area, Oka did a great deal of works. Oka [9] obtained $\beta_{tp/j} \in \pi_*(S^o)_p$ for $p \geqslant 5, t \geqslant 1, 1 \leqslant j \leqslant p-1; j \leqslant p$ if $t \geqslant 2$. [10] also obtained $\beta_{tp^2/j} \in \pi_*(S^o)_p$ for $t \geqslant 1$, $1 \leqslant j \leqslant 2p-2$; $j \leqslant 2p$ if $t \geqslant 2$. [11] [12] further generalized the above results to $\beta_{tp^n/j} \in \pi_*(S^0)_p$ for $n \geqslant 2, t \geqslant 1, 1 \leqslant j \leqslant 2^{n-1}(p-1); j \leqslant 2^{n-1}p$ if $t \geqslant 2$.

These are obtained on the basis of deep research on the structure of graded ring $[\Sigma^* V_j; V_j]$, where V_j is a 4 cells spectrum such that $BP_*(V_j) = BP_*/(p, v_1^j)$. [11] [12] showed that V_j is a split commutative ring spectrum if $j \equiv 0 \pmod p$ and is a commutative ring spectrum but nonsplit if $j \not\equiv 0 \pmod p$. In both cases, the structure of $[\Sigma^* V_j; V_j]$ differs in the two cases. [11] obtained the above $\beta_{tp^n/j}$ for $t \geqslant 2$ by using the structure in split case and [12] obtained the above $\beta_{tp^n/j}$ for $t \geqslant 1$ by using the structure in nonsplit case.

The main purpose of the present paper is to consider some more properties of the graded ring $[\Sigma^* V_j; V_j]$ in split case and then we find that the above results obtained in [11] in case $t \geqslant 2$ can be generalized to get almost all $\beta_{tp^n/j} \in \pi_*(S^o)_p$, our main results are

THEOREM I. *For $p \geqslant 5$, $n \geqslant 1$, $t \geqslant 2$, $1 \leqslant j \leqslant p^n$, the generators*

$$\beta_{tp^n/j} \in \mathrm{Ext}^{2,*}_{BP_*BP}(BP_*, BP_*)$$

survive nontrivially to $\pi_*(S^0)_p$ in the Adams–Novikov spectral sequence and the corresponding homotopy classes are of p order and are linearly independent.

THEOREM II. *If $1 \leqslant j \leqslant p^{n-1}$ and $j \equiv 0 \pmod p$, the above $\beta_{tp^n/j} \in \pi_*(S^0)_p$ are divisible by p.*

THEOREM III. *If p, n, t, j are as those in the Theorem I, then there exists a spectrum X such that $BP_*(X) = BP_*/(p, v_1^j, v_2^{tp^n})$.*

Theorem I is proved by double induction. The input of the first induction is a result of Oka [9] that there exists a self map ξ_n: $\Sigma^{2tp^n}(p^2-1)\ V_{p^n} \to V_{p^n}$ such that $(\xi_n)_* = v_2^{tp^n}$ if $n = 1$. It follows from a result in [10] that $h_1 \xi_n^p = \xi_n^p h_1$, where h_{k-1} is a map in the following cofibration (write $q = 2(p-1)$)

$$\Sigma^{(k-1)p^n q}\ V_{p^n} \to V_{kp^n} \to V_{(k-1)p^n} \xrightarrow{h_{k-1}} \Sigma^{(k-1)p^n q+1}\ V_{p^n}$$

Write $\xi_n^p = f_1$, then $f_1 \in [\Sigma^{tp^{n+1}(p+1)q}\ V_{p^n};\ V_{p^n}]$ such that $(f_1)_* = v_2^{tp^{n+1}}$. It follows from the property of cofibration that there exists $f_2 \in [\Sigma^{tp^{n+1}(p+1)q}\ V_{2p^n};\ V_{2p^n}]$ such that $(f_2)_* = v_2^{tp^{n+1}}$ and then start the second induction to get $f_k \in [\Sigma^{tp^{n+1}(p+1)q}\ V_{kp^n};\ V_{kp^n}]$ such that $(f_k)_* = v_2^{tp^{n+1}}$ for $1 \leqslant k < p$. The second induction is done by using BP method and the properties of split ring spectra V_{kp^n} and it can be done only if $k < p$. By using some results on invariant ideals in BP_*, we can then prove the existence of ξ_{n+1} to complete the first induction.

§2. SOME PRELIMINARIES ON BP

Let BP denote Brown–Peterson spectrum p localized, p a prime. It is well known that $BP_* = \pi_*(BP) = Z_{(p)}[v_1, v_2, \dots]$ and $BP_*BP = BP_*[t_1, t_2, \dots]$, where the generators v_i, t_i have degree $2(p^i-1)$. An ideal $I \subset BP_*$ is called invariant if $I \cdot BP_*BP = BP_*BP \cdot I$, $a \in BP_*$ is called mod I invariant if $\eta_R a = \eta_L a$ (mod $I \cdot BP_*BP$), where η_R, η_L: $BP_* \to BP_*BP$ are right and left unit respectively. Landweber [4] proved that the only prime invariant ideal is $I_n = (p, v_1, \dots, v_{n-1})$ $(1 \leqslant n \leqslant \infty)$.

For $x \in BP_*$, invariant ideal $I \subset J$ of BP_*, the multiplication by x induces a BP_* homomorphism $BP_*/I \to BP_*/J$, $y \bmod I \mapsto xy \bmod J$, which is also denoted by x. If $x = 1$, it is called canonical projection which is denoted by ρ_*. From [4], BP_*/I_n is a BP_*BP comodule and we have

$$\mathrm{Ext}^{0,*}_{BP_*BP}(BP_*, BP_*/I_n) = Z_p[v_n]$$

Miller and Wilson [6] showed that (I_n, v_n^j, y) is an invariant ideal if and only if $\bar{y} \in \mathrm{Ext}^{0,*}_{BP_*BP}(BP_*, BP_*/(I_n, v_n^j))$, where $\bar{y} = y \bmod (I_n, v_n^j)$. In the following, we give the description of the above $\bar{y}$ given in [6] only in the $n = 1$ case.

PROPOSITION 2.1. ([6] p. 140 Proposition 6.3 in case $n = 1$) *Let $p > 2$, then $\mathrm{Ext}^{0,*}_{BP_*BP}(BP_*, BP_*/(p, v_1^j))$ is a $Z_p[v_1]/(v_1^j)$ submodule of $BP_*/(p, v_1^j)$ generated by* 1 *and $v_1^m \bar{c}_1(r)$, where $m = \max\{0, j - q_1(r)\}$ $r = ap^s$, $a \not\equiv 0$ (mod p) and* (cf. p. 137)

$$q_1(ap^s) = \begin{cases} p^s & \text{if } a = 1 \\ p^s + p^{s-1} - 1 & \text{otherwise} \end{cases}$$

$$\bar{c}_1(ap^s) = v_2^{ap^s} \qquad \text{if } a = 1 \text{ or } s = 0$$

and for otherwise

$$\bar{c}_1(ap^s) = v_2^{ap^s} - av_1^{b_1} v_2^{a_1 p^{s-2}} - 2a \sum_{j=2}^{s-1} v_1^{b_j} v_2^{a_j p^{s-1-j}}$$

modulo $(p, v_1^{p^s})$, *where* $a_j = (ap-1)p^j + 1$, $b_j = p^s + p^{s-1} - p^{s-j} - p^{s-j-1}$ $(j \geqslant 1)$.

The following corollary is well known; the proof is omitted.

COROLLARY 2.2. *If* $j \leqslant p^s$, $v_2^{ap^s} \in \mathrm{Ext}^{o,*}_{BP_*BP}(BP_*, BP_*/(p, v_1^j))$.

The following corollary will play an important role in the proof of the main Theorem I.

COROLLARY 2.3. *Let* $p > 2$, $x \in BP_*$ *such that* $|v_1^{(k-1)p^n} x| = tp^{n+1}(p+1)q$, *where* $q = 2(p-1)$ *and* $t \geqslant 1$, $n \geqslant 0$, *then*

(1) *If* $2 \leqslant k < p$, $v_1^{(k-1)p^n} x = 0$ *in* $\mathrm{Ext}^{o,*}_{BP_*BP}(BP_*, BP_*/(p, v_1^{kp^n}))$.

(2) *If* $k = p$, $v_1^{(k-1)p^n} x = \sum_{j=1}^{[n+1/2]} \lambda_j v_1^{p^{n+1} - p^{n+1-2j}} \bar{c}_1(a_j p^{n+1-2j})$ *in* $\mathrm{Ext}^{o,*}_{BP_*BP}(BP_*, BP_*/(p, v_1^{p^{n+1}}))$, *where* $\lambda_j \in Z_p$, $a_j = tp^{2j} - \dfrac{p^{2j-1}}{p-1}$.

Proof. (1) Proposition 2.1 shows that $v_1^{(k-1)p^n} x$ must be a linear combination of elements $v_1^r \bar{c}_1(ap^s)$, where $(k-1)p^n \leqslant r < kp^n$ and $r \geqslant m = \{0,\ kp^n - q_1(ap^s)\} \geqslant kp^n - p^s - p^{s-1} + 1$. For degree reason we have

$$rq + ap^s(p+1)q = tp^{n+1}(p+1)q$$

Hence $r = (tp^{n+1} - ap^s)(p+1)$ and so r is divisible by p^s, we have p-adic expansion of r as follows

$$r = (k-1)p^n + b_{n-1}p^{n-1} + \dots + b_s p^s$$

where $0 \leqslant b_i < p$. On the other hand r is also divisible by $p+1$, then

$$(k-1) - b_{n-1} + b_{n-2} - \dots + (-1)^{n-s} b_s = 0$$

and so b_i can't be all equal to $p-1$, i.e. there is $j \geqslant s$ such that

$$\begin{aligned} b_{n-1}p^{n-1} + \dots + b_s p^s + p^s + p^{s-1} - 1 &\leqslant (p-1)(p^{n-1} + \dots + p^s) - p^j \\ &\quad + p^s + p^{s-1} - 1 = p^n - p^s - p^j + p^s + p^{s-1} - 1 < p^n \end{aligned}$$

then $r < kp^n - p^s - p^{s-1} + 1 \leqslant m$, yields contradiction.

(2) If $k = p$, it follows from $r = (p-1)p^n + b_{n-1}p^{n-1} + \dots + b_s p^s \geqslant m \geqslant p^{n+1} - p^s - p^{s-1} + 1$ that $b_{n-1}p^{n-1} + \dots + b_s p^s + p^s + p^{s-1} - 1 \geqslant p^n$ and so $b_{n-1}, b_{n-2}, \dots, b_s$ are all equal to $p-1$, and it follows from the following equation

$$(p-1) - b_{n-1} + b_{n-2} - \dots + (-1)^{n-s} b_s = 0$$

that $n-s$ must be odd. Let $n-s = 2j-1$ $(j \geqslant 1)$, then $s = n+1-2j$ and $j \leqslant [n+1/2]$, $r = (p-1)p^n + (p-1)p^{n-1} + \dots + (p-1)p^{n+1-2j} = p^{n+1} - p^{n+1-2j}$ and from $r = p^{n+1} - p^{n+1-2j} = (tp^{n+1} - a_j p^{n+1-2j})(p+1)$, we have $a_j = tp^{2j} - (p^{2j}-1)/(p-1)$, the proposition is proved.

COROLLARY 2.4. *If* $2 \leqslant k < p$, *then*

$$\mathrm{Ext}^{o,\,tp^{n+1}(p+1)q - (k-1)p^n q}_{BP_*BP}(BP_*, BP_*/(p, v_1^{p^n})) = 0$$

Proof. If there is a generator $v_1^b \bar{c}_1(ap^s)$, from 2.1 we have

$$p^n > b \geqslant m = \max\{0, p^n - q_1(ap^s)\} \geqslant p^n - p^s - p^{s-1} + 1$$

and for degree reason, $bq + ap^s(p+1)q = tp^{n+1}(p+1)q - (k-1)p^n q$. Let $r = (k-1)p^n + b$, then $rq + ap^s(p+1)q = tp^{n+1}(p+1)q$ and $(k-1)p^n \leqslant r < kp^n$, $r \geqslant kp^n - p^s - p^{s-1} + 1$. Similar to the proof of 2.3, there exists no such b in the case $2 \leqslant k < p$.

For $t \geqslant 1$, [7] defined $\beta_{tp^n/j} = \delta\delta_j v_2^{tp^n} \in \mathrm{Ext}^{2,*}_{BP_*BP}(BP_*, BP_*)$, where δ, δ_j are the boundary homomorphisms in the Ext sequences induced by the following two short exact sequences respectively

$$E\colon\ 0 \to BP_* \xrightarrow{p} BP_* \to BP_*/(p) \to 0$$

$$E_j\colon\ 0 \to BP_*/(p) \xrightarrow{v_1^j} BP_*/(p) \to BP_*/(p, v_1^j) \to 0$$

[7] proved that if $1 \leqslant j \leqslant p^n$, $\beta_{tp^n/j} \neq 0$, are of p order and linearly independent, having internal degree $2tp^n(p^2-1) - 2j(p-1)$.

The following proposition guarantees that if there exists a self map $f\colon \Sigma^{tp^n(p+1)q} V_j \to V_j$ such that $f_* = v_2^{tp^n}$, then $\beta_{tp^n/j}$ survives nontrivially to $\pi_*(S^o)_p$ in the Adams–Novikov spectral sequence, where V_j is a spectrum such that $BP_* V_j = BP_*/(p, v_1^j)$.

PROPOSITION 2.5. ([3] Theorem 1.7 or [7] Lemma 2.10). *Let $W \to X \to Y \xrightarrow{h} \Sigma W$ be a cofibration of finite spectra such that the induced BP_* homomorphism $h_* = 0$, δ: $\mathrm{Ext}^s(BP_*, BP_* Y) \to \mathrm{Ext}^{s+1}(BP_*;\ BP_* W)$ be boundary homomorphism in* Ext *exact sequence. If $\bar{x} \in \mathrm{Ext}^s(BP_*, BP_* Y)$ survives to $x \in \pi_*(Y)_p$ in the Adams–Novikov spectral sequence, then $\delta\bar{x}$ survives to $h_*(x) \in \pi_*(W)_p$.*

§3. SPLIT RING SPECTRA AND ITS PROPERTIES

In this section, we recall some results on the structure of graded ring $[\Sigma^* K; K]$ of split ring spectra K over Z_t given in [11]. Moreover, we consider further properties of split ring spectra V_j over Z_p which are not in [11].

Let M_t (or briefly M) $= S^o \cup_t e^1$ be Moore spectrum over the cyclic group Z_t, then there is a cofibration

$$S^o \xrightarrow{t} S^o \xrightarrow{j_o} M \xrightarrow{\tau} \Sigma S^o$$

Since $t \cdot 1_M$ is zero in $[M; M]$, then $M \wedge M = M \vee \Sigma M$. Let $\mu_M\colon M \wedge M \to M$ be the projection to the first summand and $\bar{\mu}_M\colon \Sigma M \to M \wedge M$ be injection, then define

$$d_M\colon [\Sigma^r M; M] \to [\Sigma^{r+1} M; M]$$

to be $d_M(f) = \mu_M(1_M \wedge f)\bar{\mu}_M$, proposition 1.1 in [11] showed that d_M is a derivation, i.e. $d_M^2 = 0$ and

$$d_M(gf) = (-1)^k d_M(g) f + g d_M(f)$$

where $g \in [\Sigma^* M; M]$, $f \in [\Sigma^k M; M]$. Let $\delta_M = j_0 \tau\colon M \to \Sigma M$ then $d_M(\delta_M) = -1_M$ (cf. [11] p. 272) and

$$[\Sigma^* M; M] = (\mu[\Sigma^* M; M] \oplus \mu[\Sigma^* M; M] \cdot \delta_M$$

where $\mu[\Sigma^* M; M] = \ker d_M \cap [\Sigma^* M; M]$. [11] showed that M is a commutative ring spectrum with unique multiplication $\mu_M\colon M \wedge M \to M$.

the cofibre of map $\phi:\Sigma^k M \to M$, i.e. there is a cofibration

$$\Sigma^k M \xrightarrow{\phi} M \xrightarrow{i} K \xrightarrow{\pi} \Sigma^{k+1} M$$

cell decomposition $K = S^o \cup e^1 \cup e^{k+1} \cup e^{k+2}$, to have a decomposition $\vee \Sigma K \vee \Sigma^{k+1} K \vee \Sigma^{k+2} K$, we introduce

3.1. ([11] Definition 2.1) K is said to be a split ring spectrum (over Z_t) if $K; K]$ and $\phi \wedge 1_K = 0$ in $[\Sigma^k M \wedge K; M \wedge K]$.
condition provides us with decomposition $M \wedge K = K \vee \Sigma K$ and the second provides $K \wedge K = (M \wedge K) \vee \Sigma^{k+1}(M \wedge K)$, then $K \wedge K$ splits into $\Sigma^{k+1} K \vee \Sigma^{k+2} K$. Let

$$\begin{array}{ll} \mu_1: K \wedge K \to K, & \nu_1: K \to K \wedge K \\ \mu_2: K \wedge K \to \Sigma K & \nu_2: \Sigma K \to K \wedge K \\ \mu_3: K \wedge K \to \Sigma^{k+1} K, & \nu_3: \Sigma^{k+1} K \to K \wedge K \\ \mu_4: K \wedge K \to \Sigma^{k+1} K, & \nu_4: \Sigma^{k+2} K \to K \wedge K \end{array} \tag{3.2}$$

ns to each summand and injections from each summand, then $\mu_i \nu_i = 1_K$, j). Let d_K, d'_K (or briefly d, d')

$$\begin{aligned} d_K&: [\Sigma^r K; K] \to [\Sigma^{r+1} K; K] \\ d'_K&: [\Sigma^r K; K] \to [\Sigma^{r+k+1} K; K] \end{aligned} \tag{3.3}$$

o be $d_K(f) = \mu_1(1_K \wedge f)\nu_2$, $d'_K(f) = \mu_1(1_k \wedge f)\nu_3$, then d_K, d'_K are two "deriva- $^*K; K]$, i.e. they satisfy

$$\begin{aligned} &fg) = (-1)^j d_K(f)g + f d_K(g) \\ &fg) = (-1)^j d'_K(f)g + f d'_K(g) + d_K(f)(\phi' \wedge 1_K) d_K(g) \\ &d_K^2 = 0,\ d_K d'_K = -d'_K d_K \quad \text{and} \quad (d'_K)^2 = 0 \text{ if } t \not\equiv 0 \pmod 3 \end{aligned} \tag{3.4}$$

$\Sigma^* K; K]$ and $\deg g = j$, ϕ' is an element of $\pi_*(S^o)_p$ such that $t\phi'$ is an attaching :f. [11] p 269).
r, there exist two elements $\delta_K \in [\Sigma^{-1} K; K]$, $\delta'_K \in [\Sigma^{-k-1} K; K]$ (or briefly ng

$$\begin{array}{llll} d(\delta) = -1_K, & d'(\delta) = 0, & d(\delta') = 0, & d'(\delta') = -1_K \\ (\delta')^2 = 0, & \delta\delta' = -\delta'\delta, & \delta^2 = \delta'(\phi' \wedge 1) & \\ \delta_i = i\delta_M, & \pi\delta = -\delta_M \pi & & \end{array} \tag{3.5}$$

$= -1_K$ only holds in case $t \not\equiv \pm 3 \bmod 9$ (cf. [11] p. 270 and Lemma 2.6) and

$$\delta' = i\pi: K \xrightarrow{\pi} \Sigma^{k+1} M \xrightarrow{i} \Sigma^{k+1} K \tag{3.6}$$

78 (2.3)).
"derivations" d_K, d'_K make the graded ring $[\Sigma^* K; K]$ to have the following cture. Let $\mathscr{C}_*(K) = \ker d \cap \ker d'$ be a subgroup of $[\Sigma^* K; K]$, then we have

3.7. ([11] *Theorem* 5.5) *If* $t \not\equiv 0 \pmod 3$, $\mathscr{C}_*$ *is a summand of* $[\Sigma^* K; K]$) *which the direct sums in the following* 16 *ways*

$$[\Sigma^* K; K] = \mathscr{C}_* \oplus \mathscr{D}_* \oplus \mathscr{D}'_* \oplus \mathscr{D}''_*$$

for
$$\mathscr{D}_* = \mathscr{C}_*\delta, \delta\mathscr{C}_*$$
$$\mathscr{D}'_* = \mathscr{C}_*\delta', \delta'\mathscr{C}_*$$
$$\mathscr{D}''_* = \mathscr{C}_*\delta\delta', \mathscr{C}_*\delta'\delta, \delta'\mathscr{C}_*\delta, \delta\delta'\mathscr{C}_*$$

Moreover, [11] Theorem 5.6 and Corollary 5.7 showed that $\mathscr{C}_*$ is a commutative subring of $[\Sigma^*K; K]$ and if $t \not\equiv 0 \pmod 3$, $f \in \mathscr{C}_*$ having even degree, then $[\delta', f^t] = [\delta, f^t] = 0$, where $[\ ,\ ]$ denotes the graded commutator, i.e. $[f, f'] = ff' - (-1)^{|f|\cdot|f'|}f'f$, then we have

$$\delta' f^t = f^t \delta' \tag{3.8}$$
$$\delta f^t = f^t \delta \qquad \text{if } f \text{ has even degree}, f \in \mathscr{C}_* \tag{3.8}$$

THEOREM 3.10. *If $t \not\equiv 0 \pmod 3$, split ring spectrum K is an associative commutative ring spectrum with multiplication μ_1.* (cf. [11] Theorem 4.13)

The following is a sufficient condition of the cofibre K of ϕ to be a split ring spectrum.

PROPOSITION 3.11. ([11] Proposition 2.9). *If $\phi = f^n (n \geqslant 2)$ and $d_M(f) = 0$, where $f \in [\Sigma^*M; M]$ having even degree, then the cofibre K of ϕ is a ring spectrum. Moreover, K is a split ring spectrum if $n \equiv 0 \pmod t$.*

The following are examples of split ring spectra given in [11], and will also be used in the proof of the main Theorems I and II.

Let $M(m) = M_{p^m} = S^0 \cup_{p^m} e^1$ be Moore spectrum over the cyclic group Z_{p^m}, we have

THEOREM 3.12. ([11] Theorem 6.2) *There exists a self map*
$$A_m: \Sigma^{2p^{m-1}(p-1)}M(m) \to M(m), \quad m \geqslant 1$$
such that $d_M(A_m) = 0$ and the induced BP_* homomorphism
$$(A_m)_* = v_1^{p^{m-1}}: BP_*/(p^m) = BP_*/(p^m)$$
Let $M(m, sp^{m-1})$ be the cofibre of A_m^s, i.e. there is a cofibration
$$\Sigma^{sp^{m-1}q}M(m) \xrightarrow{A_m^s} M(m) \to M(m, sp^{m-1})$$
realized the following short exact sequence
$$0 \to BP_*/(p^m) \xrightarrow{v_1^{sp^{m-1}}} BP_*/(p^m) \to BP_*/(p^m, v_1^{sp^{m-1}}) \to 0$$
i.e. $BP_*M(m, sp^{m-1}) = BP_*/(p^m, v_1^{sp^{m-1}})$, then the following corollary follows from Proposition 3.11 and Theorem 3.12 directly.

COROLLARY 3.13. *If $s \equiv 0 \pmod{p^m}$, $M(m, sp^{m-1})$ is a split ring spectrum over Z_{p^m}.*

In this paper, we only use $M(m, sp^{m-1})$ for $m = 1$, $s = kp^n$ $(n \geqslant 1)$. Write $V(0) = M(1) = S^0 \cup_p e^1$ be Moore spectrum over Z_p, $\phi = A_1$: $\Sigma^q V(0) \to V(0)$, $V_{kp^n} = M(1, kp^n)$, then there is a cofibration
$$\Sigma^{kp^nq} V(0) \xrightarrow{\phi^{kp^n}} V(0) \xrightarrow{i_k} V_{kp^n} \xrightarrow{\pi_k} \Sigma^{kp^nq+1} V(0) \tag{3.14}$$

realized the following short exact sequence

$$0 \to BP_*/(p) \xrightarrow{v_1^{kp^n}} BP_*/(p) \to BP_*/(p, v_1^{kp^n}) \to 0$$

Then it follows from 3.13 that if $n \geqslant 1$, V_{kp^n} is a split ring spectrum over Z_p.

Write δ' in (3.6) as $\bar{h}_k$, i.e.

$$\bar{h}_k = i_k \pi_k : V_{kp^n} \xrightarrow{\pi_k} \Sigma^{kp^n q+1} V(0) \xrightarrow{i_k} \Sigma^{kp^n q+1} V_{kp^n} \tag{3.15}$$

and δ in (3.5) still write as δ, then $[\Sigma^* V_{kp^n}; V_{kp^n}]$ has two "derivations" d and d' and from (3.5), (3.4) we have

$$\begin{aligned} &d(\bar{h}_k) = 0,\ d'(\bar{h}_k) = -1_{V_{kp^n}},\ d(\delta) = -1_{V_{kp^n}}, d'(\delta) = 0 \\ &\delta i_k = i_k \delta_{V(0)},\ \pi_k \delta = -\delta_{V(0)} \pi_k \\ &\mathrm{d}(fg) = (-1)^j d(f) g + f d(g) \\ &d'(\bar{h}_k g) = (-1)^j d'(\bar{h}_k) g + \bar{h}_k d'(g) \\ &d'(f \bar{h}_k) = (-1)^t d'(f) \bar{h}_k + f d'(\bar{h}_k) \end{aligned} \tag{3.16}$$

where $j = \deg g$ and write $\deg \bar{h}_k$ as t which is $= -kp^n q - 1$.

Since in this paper we only consider p as a prime $\geqslant 5$, then $p \not\equiv 0 \pmod 3$ and so, from (3.4) we have

$$(d')^2 = 0, \quad d^2 = 0, \quad dd' = -d'd \tag{3.17}$$

Let $\mathscr{C}_*(V_{kp^n}) = \ker d \cap \ker d'$, from Theorem 3.7 we have

$$\begin{aligned} [\Sigma^* V_{kp^n}; V_{kp^n}] &= \mathscr{C}_* \oplus \delta\mathscr{C}_* \oplus \bar{h}_k \mathscr{C}_* \oplus \bar{h}_k \delta \mathscr{C}_* \\ &= \mathscr{C}_* \oplus \mathscr{C}_* \delta \oplus \mathscr{C}_* \bar{h}_k \oplus \mathscr{C}_* \delta \bar{h}_k \end{aligned} \tag{3.18}$$

and $\mathscr{C}_*(V_{kp^n})$ is a commutative subring. For $f \in \mathscr{C}_*$ having even degree we have

$$\bar{h}_k f^p = f^p \bar{h}_k, \qquad \delta f^p = f^p \delta \tag{3.19}$$

Now the recollection of result on split ring spectra given in [11] is concluded and we turn to consider some properties of the split ring spectrum V_{kp^n} over Z_p which are not in [11] and are important in the proof of the main theorem.

PROPOSITION 3.20. *If $r \geqslant k \geqslant 1$, $m \geqslant n \geqslant 1$*

$$\begin{aligned} &(i_r)_* : [\Sigma^t V_{kp^n}; V(0)] \to [\Sigma^t V_{kp^n}; V_{rp^m}] \\ &(\pi_r)^* : [\Sigma^t V(0); V_{kp^n}] \to [\Sigma^{t - rp^m q - 1} V_{rp^m}; V_{kp^n}] \end{aligned}$$

is monic, where i_r, π_r cf. (3.15).

In order to prove the proposition, we need the following lemma which is obvious for dimensional reason.

LEMMA (1) *If $t < -rp^m q - 2$, $[\Sigma^t V_{rp^m}; V(0)] = 0$*

(2) *If $t < -1$, $[\Sigma^t(V(0); V_{rp^m}] = 0$.*

Proof of Prop. 3.20. For $g \in [\Sigma^t V_{kp^n}; V(0)]$ such that $i_r g = 0$, there exists g_1 such that $g = \phi^{rp^m} g_1$ as in the following diagram

$$\begin{tikzcd} \Sigma^{rp^mq} V(0) \arrow[r, "\phi^{rp^m}"] & V(0) \arrow[r, "i_r"] & V_{rp^m} \\ & \Sigma^t V_{kp^n} \arrow[ul, "g_1"] \arrow[u, "g"'] & \end{tikzcd}$$

Since $\bar{h}_k i_k g_1 = i_k \pi_k i_k g_1 = 0$, then by acting "derivation" d' (cf. 3.16) we have

$$0 = d'(\bar{h}_k i_k g_1) = \pm d'(\bar{h}_k) i_k g_1 + \bar{h}_k d'(i_k g_1)$$

and recall that $d'(\bar{h}_k) = -1$, then

$$i_k(\pm g_1 + \pi_k d'(i_k g_1)) = 0$$

and so there exists $g_2 \in [\Sigma^t V_{kp^n}; \Sigma^{rp^mq + kp^nq} V(0)]$ such that

$$\pm g_1 + \pi_k d'(i_k g_1) = \phi^{kp^n} g_2$$

Hence $g = \phi^{rp^m} g_1 = \pm \phi^{rp^m + kp^n} g_2$ (since $r \geqslant k$, $m \geqslant n$ we have $\phi^{rp^m} \pi_k = 0$). Repeating the above process, then there exists $g_{s+1} \in [\Sigma^t V_{kp^n}; \Sigma^{rp^mq + skp^nq} V(0)]$ such that $g = \pm \phi^{rp^m + skp^n} g_{s+1}$. But it follows from Lemma 3.21 that $g_{s+1} = 0$ for s large, so $g = 0$.

The proof of the second half is similar.

PROPOSITION 3.22. *If* $k \geqslant 1$, $n \geqslant 1$

$$(i_k)^* : im d' \to [\Sigma^* V(0); V_{kp^n}]$$
$$(\pi_k)_* : im d' \to [\Sigma^* V_{kp^n}; V(0)]$$

is epic.

Proof. For any $g \in [\Sigma^t V(0); V_{kp^n}]$, $g \pi_k \bar{h}_k = 0$ and from (3.16)

$$0 = d'(g \pi_k \bar{h}_k) = -d'(g\pi_k)\bar{h}_k + g\pi_k d'(\bar{h}_k)$$

Then $(-d'(g\pi_k) i_k - g)\pi_k = 0$ and there exists $g_1 \in [\Sigma^{t - kp^nq} V(0); V_{kp^n}]$ such that $g + d'(g\pi_k) i_k = g_1 \phi^{kp^n}$. Suppose that for $s \geqslant 1$ there exists $g_s \in [\Sigma^{t - skp^nq} V(0); V_{kp^n}]$ such that $g + d'(g\pi_k) i_k = g_s \phi^{skp^n}$, then $0 = d'(g_s \pi_k \bar{h}_k) = -d'(g_s\pi_k)\bar{h}_k + g_s \pi_k d'(\bar{h}_k)$ and so there exists $g_{s+1} \in [\Sigma^{t-(s+1)kp^nq} V(0); V_{kp^n}]$ such that

$$g + d'(g\pi_k) i_k = g_{s+1} \phi^{(s+1)kp^n}$$

and it follows from Lemma 3.21 that $g_{s+1} = 0$ for s large, then $g = -(i_k)^* d'(g\pi_k)$. The proof of the second half is similar.

To help prove the main theorem, we now consider some cofibrations which relate V_{kp^n} and V_{rp^n}. These cofibrations can be obtained from 3×3 Lemma (cf. [17] p.293 or [2] Lemma 6.8) in the stable homotopy category as follows.

Given a map $\gamma = \alpha\beta$: $A \xrightarrow{\beta} B \xrightarrow{\alpha} C$ of spectra, there exists a cofibration

$$C(\beta) \xrightarrow{u} C(\gamma) \xrightarrow{v} C(\alpha) \xrightarrow{w} \Sigma C(\beta)$$

such that the following diagram commutes up to homotopy and sign, where $C(\alpha)$ denotes the cofibre of α et al

$$
\begin{tikzcd}
A \arrow[rr, "\gamma"] \arrow[dr, "\beta"'] & & C \arrow[rr] \arrow[dr] & & C(\alpha) \arrow[rr] \arrow[dr] & & \Sigma C(\beta) \\
& B \arrow[ur, "\alpha"'] \arrow[dr] & & C(\gamma) \arrow[ur, "v"] \arrow[dr] & & \Sigma B \arrow[ur] \arrow[dr, "\alpha"] & \\
\Sigma^{-1} C(\alpha) \arrow[rr, "w"] \arrow[ur] & & C(\beta) \arrow[rr] \arrow[ur, "u"] & & \Sigma A \arrow[rr, "\gamma"] \arrow[ur, "\beta"] & & \Sigma C
\end{tikzcd}
$$

Recall that there is a cofibration (3.14) and let $h_{k-1} = i_{k-1}\pi_1$, then the following diagram commutes

$$
\begin{tikzcd}
V_{p^n} \arrow[rr, "h_{k-1}"] \arrow[dr, "\pi_1"'] & & \Sigma^{p^nq+1} V_{(k-1)p^n} \arrow[rr, "\pi_{k-1}"] \arrow[dr, "\psi_{k-1,k}"] & & \Sigma^{kp^nq+2} V(0) \arrow[rr, "\phi^{kp^n}"] \arrow[dr] & & \Sigma^2 V(0) \\
& \Sigma^{p^nq+1} V(0) \arrow[ur, "i_{k-1}"] \arrow[dr, "\phi^{p^n}"] & & C(h_{k-1}) \arrow[ur, "\pi_k"] \arrow[dr, "\rho_{k,1}"] & & \Sigma^{p^nq+2} V(0) \arrow[ur, "\phi^{p^n}"] \arrow[dr, "i_{k-1}"] & \\
\Sigma^{kp^nq+1} V(0) \arrow[rr, "\phi^{kp^n}"] \arrow[ur, "\phi^{(k-1)p^n}"] & & \Sigma V(0) \arrow[rr, "i_1"] \arrow[ur, "i_k"] & & \Sigma V_{p^n} \arrow[rr, "h_{k-1}"] \arrow[ur, "\pi_1"] & & \Sigma^{p^nq+2} V_{(k-1)p^n}
\end{tikzcd}
$$

From the above 3×3 lemma, we know that in the diagram $C(h_{k-1}) = \Sigma C(\phi^{kp^n}) = \Sigma V_{kp^n}$. Since the induced BP_* homomorphism $(\phi^{p^n})_* = v_1^{p^n}$, then $(\psi_{k-1,k})_* = v_1^{p^n}$, and the diagram commutes up to positive sign, then we have the first fundamental cofibration in this paper

$$V_{p^n} \xrightarrow{h_{k-1}} \Sigma^{p^nq+1} V_{(k-1)p^n} \xrightarrow{\psi_{k-1,k}} \Sigma V_{kp^n} \xrightarrow{\rho_{k,1}} \Sigma V_{p^n}$$

which realizes the following short exact sequence

$$0 \to BP_*/(p, v_1^{(k-1)p^n}) \xrightarrow{v_1^{p^n}} BP_*/(p, v_1^{kp^n}) \to BP_*/(p, v_1^{p^n}) \to 0$$

and also have the following relations

$$\psi_{k-1,k} i_{k-1} = i_k \phi^{p^n} \tag{3.23}$$

$$\rho_{k,1} i_k = i_1 \tag{3.24}$$

$$\pi_{k-1} = \pi_k \psi_{k-1,k} \tag{3.25}$$

$$\pi_1 \rho_{k,1} = \phi^{(k-1)p^n} \pi_k \tag{3.26}$$

Let $h'_{k-1} = i_1 \pi_{k-1}$, similarly we have the second fundamental cofibration as follows

$$V_{(k-1)p^n} \xrightarrow{h'_{k-1}} \Sigma^{(k-1)p^nq+1} V_{p^n} \xrightarrow{\psi_{1,k}} \Sigma V_{kp^n} \xrightarrow{\rho_{k,k-1}} \Sigma V_{(k-1)p^n}$$

which realizes a similar BP_* short exact sequence such that $(\psi_{1,k})_* = v_1^{(k-1)p^n}$ and also have the following relations

$$\psi_{1,k} i_1 = i_k \phi^{(k-1)p^n} \tag{3.27}$$

$$\rho_{k,k-1} i_k = i_{k-1} \tag{3.28}$$

$$\pi_1 = \pi_k \psi_{1,k} \tag{3.29}$$

$$\pi_{k-1} \rho_{k,k-1} = \phi^{p^n} \pi_k \tag{3.30}$$

Let $\gamma = \psi_{k-1,k}\psi_{1,k-1}$, then we also have 3×3 diagram as follows

$$\begin{tikzcd}
\Sigma^{(k-1)p^n q} V_{p^n} \arrow[rr, "\gamma"] \arrow[dr, "\psi_{1,k-1}"'] & & V_{kp^n} \arrow[rr, "\rho_{k,1}"] \arrow[dr, "\bar{\rho}"] & & V_{p^n} \\
& \Sigma^{p^n q} V_{(k-1)p^n} \arrow[ur, "\psi_{k-1,k}"] \arrow[dr, "\rho_{k-1,k-2}"] & & C(\gamma) \arrow[ur, "v = \rho_{k-1,1}"'] \arrow[dr, "\bar{h}"] & \\
\Sigma^{-1} V_{p^n} \arrow[ur, "h_{k-1}"] \arrow[rr, "w = h_{k-2}"] & & \Sigma^{p^n q} V_{(k-2)p^n} \arrow[ur, "u = \psi_{k-2,k-1}"] \arrow[rr, "h'_{k-2}"] & & \Sigma^{kp^n q+1} V_{p^n}
\end{tikzcd}$$

It is easily seen that $C(\gamma) = V_{(k-1)p^n}$ and $\bar{h}\psi_{k-2,k-1} = h'_{k-2} = i_1\pi_{k-1}\psi_{k-2,k-1} = h'_{k-1}\psi_{k-2,k-1}$. Hence we have

$$\bar{h} = h'_{k-1} + \eta\rho_{k-1,1}$$

for some $\eta \in [\Sigma^{-(k-1)p^n q-1} V_{p^n}; V_{p^n}]$, and it is easily seen by degree reason that $\eta = 0$ since $k \geqslant 3$. Therefore, $\bar{h} = h'_{k-1}$ and so $\gamma = \psi_{1,k}$ and $\bar{\rho} = \rho_{k,k-1}$. Then we have

$$\psi_{1,k} = \psi_{k-1,k}\psi_{1,k-1} = \psi_{k-1,k} \cdots \psi_{2,3}\psi_{1,2} \tag{3.31}$$

$$\rho_{k,1} = \rho_{k-1,1}\,\rho_{k,k-1} \tag{3.32}$$

The following relations also can be easily obtained

$$\pi_{k-j} = \pi_k\psi_{k-j,k} \tag{3.33}$$

$$\pi_j\rho_{k,j} = \phi^{(k-1)p^n}\pi_k \qquad 1 \leqslant j < k \tag{3.34}$$

The above relations (3.23)–(3.34) will be used later.

§4. AN INDUCTION. PROOF OF THE MAIN THEOREM

The main theorem will be proved by double induction. The input of the first induction is the following result given in S. Oka [10].

THEOREM 4.1. ([10] Lemma 4.2). *If $t \geqslant 2$, there exists a self map*

$$\xi : \Sigma^{tp(p+1)q} V_p \to V_p$$

such that $\xi_* = v_2^{tp} : BP_*/(p, v_1^p) \to BP_*/(p, v_1^p)$ and $\xi \in \mathscr{C}_*(V_p)$.

For the first induction, suppose that for $1 \leqslant r \leqslant n$ there exists

$$\xi_r : \Sigma^{tp^r(p+1)q} V_{p^r} \to V_{p^r} \text{ such that } (\xi_r)^* = v_2^{tp^r} \text{ and } \xi_r \in \mathscr{C}_*(V_{p^r}).$$

Let $f_1 = (\xi_n)^p$, then $(f_1)_* = v_2^{tp^{n+1}}$. In the second induction, we prove that for $1 \leqslant k < p$ there exists $f_k \in [\Sigma^{tp^{n+1}(p+1)q} V_{kp^n}; V_{kp^n}]$ such that $(f_k)_* = v_2^{tp^{n+1}}$, and further prove the existence of ξ_{n+1} to complete the first induction.

Now we proceed to prove Theorem I. Since $\xi_n \in \mathscr{C}_*(V_{p^n})$, then it follows from (3.19) that $\bar{h}_1\xi_n^p = \xi_n^p\bar{h}_1$, that is, $\bar{h}_1 f_1 = f_1\bar{h}_1$, then there exists g_2 such that the following diagram commutes

$$\begin{tikzcd}
V_{p^n} \arrow[r, "\bar{h}_1"] & \Sigma^{p^n q+1} V_{p^n} \arrow[r, "\psi_{1,2}"] & \Sigma V_{2p^n} \arrow[r, "\rho_{2,1}"] & \Sigma V_{p^n} \\
\Sigma^m V_{p^n} \arrow[u, "f_1"] \arrow[r, "\bar{h}_1"] & \Sigma^{m+p^n q+1} V_{p^n} \arrow[u, "f_1"] \arrow[r, "\psi_{1,2}"] & \Sigma^{m+1} V_{2p^n} \arrow[u, "g_2"] \arrow[r, "\rho_{2,1}"] & \Sigma^{m+1} V_{p^n} \arrow[u, "f_1"]
\end{tikzcd} \tag{4.2}$$

where $m = tp^{n+1}(p+1)q$. Since $(f_1)_* = v_2^{tp^{n+1}}$, then $(g_2 i_0)_* = v_2^{tp^{n+1}} + v_1^{p^n} x$, then it follows from Corollary 2.3(1) that $(g_2)_* = v_2^{tp^{n+1}}$. Recall that $h_k = i_k \pi_1$, $h'_k = i_1 \pi_k$, we first prove the following

LEMMA 4.3. *There exists* $f_2 \in [\Sigma^m V_{2p^n}; V_{2p^n}]$ *such that* $(f_2)_* = v_2^{tp^{n+1}}$ *and* $h_2 f_1 = f_2 h_2$, $h'_2 f_2 = f_1 h'_2$.

Proof. Since $(\hbar_2 g_2 - g_2 \hbar_2)\psi_{1,2} = i_2\pi_2 g_2 \psi_{1,2} - g_2 i_2 \pi_2 \psi_{1,2} = i_2 \pi_2 \psi_{1,2} f_1 - g_2 i_2 \pi_1 = h_2 f_1 - g_2 h_2$ and $\rho_{2,1}(h_2 f_1 - g_2 h_2) = \rho_{2,1} i_2 \pi_1 f_1 - \rho_{2,1} g_2 i_2 \pi_1 = i_1 \pi_1 f_1 - f_1 \rho_{2,1} i_2 \pi_1 = \hbar_1 f_1 - f_1 \hbar_1 = 0$, then there exists α such that the left rectangle of the following diagram commutes

$$
\begin{tikzcd}
\Sigma^{2p^nq+1} V_{p^n} \arrow[r, "\psi_{1,2}"] & \Sigma^{p^nq+1} V_{2p^n} \arrow[r, "\rho_{2,1}"] & \Sigma^{p^nq+1} V_{p^n} \arrow[r, "\hbar_1"] & \Sigma^{2p^nq+1} V_{p^n} \\
\Sigma^m V_{p^n} \arrow[u, "\alpha"] \arrow[ur, "h_2 f_1 - g_2 h_2"] \arrow[r, "\psi_{1,2}"] & \Sigma^{m-p^nq} V_{2p^n} \arrow[u, "\hbar_2 g_2 - g_2 \hbar_2"'] \arrow[r, "\rho_{2,1}"] & \Sigma^{m-p^nq} V_{p^n} \arrow[u, "\beta"'] \arrow[r, "\hbar_1"] & \Sigma^{m+1} V_{p^n} \arrow[u, "\beta"']
\end{tikzcd}
\tag{4.4}
$$

From (3.18), we can write $\alpha = \alpha_1 + \delta\alpha_2 + \hbar_1 \alpha_3 + \hbar_1 \delta \alpha_4$ such that $\alpha_i \in \mathscr{C}_*(V_{p^n})$ and so $\psi_{1,2}(\alpha_1 + \delta\alpha_2) = \psi_{1,2}\alpha = h_2 f_1 - g_2 h_2$ and $d'(\alpha_1 + \delta\alpha_2) = 0$. Hence we may assume $d'(\alpha) = 0$ and the above left rectangle still commutes. After we have had α, then there exists β such that the above diagram all commutes.

Since $\pi_1 \alpha i_1 = \pi_2 \psi_{1,2} \alpha i_1 = \pi_2(h_2 f_1 - g_2 h_2) i_1 = 0$, then $\hbar_1 \alpha \hbar_1 = 0$. Notice that α has odd degree, then it follows from (3.16) that

$$0 = d'(\hbar_1 \alpha \hbar_1) = d'(\hbar_1)\alpha\hbar_1 + \hbar_1 d'(\alpha\hbar_1) = -\alpha\hbar_1 - \hbar_1\alpha$$

then $\hbar_1 \alpha = -\alpha \hbar_1$.

On the other hand, it follows from the diagram (4.4) that

$$\pi_1 \beta \rho_{2,1} = \pi_1 \rho_{2,1}(\hbar_2 g_2 - g_2 \hbar_2) = \phi^{p^n} \pi_2 \hbar_2 g_2 - \pi_1 \rho_{2,1} g_2 \hbar_2 = -\pi_1 f_1 i_1 \pi_2$$

$$\begin{aligned}
\pi_1 \alpha \rho_{2,1} &= \pi_2 \psi_{1,2} \alpha \rho_{2,1} = \pi_2(h_2 f_1 - g_2 h_2)\rho_{2,1} = -\pi_2 g_2 i_2 \pi_1 \rho_{2,1} \\
&= -\pi_2 g_2 i_2 \phi^{p^n} \pi_2 = -\pi_2 g_2 \psi_{1,2} i_1 \pi_2 = -\pi_1 f_1 i_1 \pi_2
\end{aligned}$$

Then $\pi_1 \beta \rho_{2,1} = \pi_1 \alpha \rho_{2,1}$ and there exists $\gamma \in [\Sigma^m V_{p^n}; \Sigma^{2p^nq+1} V(0)]$ such that $\pi_1 \beta = \pi_1 \alpha + \gamma \hbar_1$. Then $\hbar_1 \beta = \hbar_1 \alpha + i_1 \gamma \hbar_1$. From diagram (4.4) we have $\hbar_1 \beta = \alpha \hbar_1$, then

$$2\alpha\hbar_1 = i_1 \gamma \hbar_1$$

By acting the "derivation" d' (using (3.16)) we have $-2\alpha = -d'(i_1\gamma)\hbar_1 - i_1\gamma$, thus $2\hbar_1\alpha = \hbar_1 d'(i_1\gamma)\hbar_1$. By acting d' again, we have

$$2\alpha = d'(i_1\gamma)\hbar_1 - \hbar_1 d'(i_1\gamma)$$

Let

$$f_2 = g_2 + \tfrac{1}{2}\psi_{1,2} d'(i_1\gamma)\rho_{2,1}$$

then

$$\begin{aligned}
f_2 h_2 &= g_2 h_2 + \tfrac{1}{2}\psi_{1,2} d'(i_1\gamma) i_1 \pi_1 \\
&= g_2 h_2 + \psi_{1,2}\alpha \\
&= g_2 h_2 + (h_2 f_1 - g_2 h_2) = h_2 f_1
\end{aligned}$$

Moreover, $(\hbar_2 f_2 - f_2 \hbar_2)\psi_{1,2} = \hbar_2 f_2 \psi_{1,2} - f_2 i_2 \pi_2 \psi_{1,2} = h_2 f_1 - f_2 h_2 = 0$, then there exists η such that the following diagram commutes

$$\begin{tikzcd}
\Sigma^{2p^nq+1} V_{p^n} \arrow[r, "\psi_{1,2}"] & \Sigma^{p^nq+1} V_{2p^n} \arrow[r, "\rho_{2,1}"] & \Sigma^{p^nq+1} V_{p^n} \arrow[r, "\bar{h}_1"] & \Sigma^{2p^nq+2} V_{p^n} \\
\Sigma^m V_{p^n} \arrow[r, "\psi_{1,2}"] \arrow[u, "0"] & \Sigma^{m-p^nq} V_{2p^n} \arrow[r, "\rho_{2,1}"] \arrow[u, "\bar{h}_2 f_2 - f_2\bar{h}_2"] & \Sigma^{m-p^nq} V_{p^n} \arrow[r, "\bar{h}_1"] \arrow[u, "\eta"] & \Sigma^{m+1} V_{p^n} \arrow[u, "0"]
\end{tikzcd}$$

By acting d' on $\bar{h}_1\eta = 0$ we have $\eta = -\bar{h}_1 d'(\eta)$. Since $\pi_1\eta i_1 = \pi_1\eta\rho_{2,1}i_2 = \pi_1\rho_{2,1}(\bar{h}_2 f_2 - f_2\bar{h}_2)i_2 = 0$, then $\bar{h}_1\eta\bar{h}_1 = 0$ and by acting d' again we have $-\eta\bar{h}_1 - \bar{h}_1\eta + \bar{h}_1 d'(\eta)\bar{h}_1 = 0$, i.e. $-2\eta\bar{h}_1 = 0$, then we have $\eta = -d'(\eta)\bar{h}_1$ and $h'_2 f_2 - f_1 h'_2 = \rho_{2,1}(\bar{h}_2 f_2 - f_2\bar{h}_2) = \eta\rho_{2,1} = 0$.

Since the induced BP_* homomorphism $(d'(i_1\gamma)i_0)_* \in \mathrm{Ext}^{0,m-p^nq}_{BP_*BP}(BP_*, BP_*/(p, v_1^{p^n})) = 0$ (cf. Cor. 2.4), where $i_o: S^o \to V_{p^n}$ is the injection of the bottom cell, then $(d'(i_1\gamma))_* = 0: BP_*/(p, v_1^{p^n}) \to BP_*/(p, v_1^{p^n})$ and so $(f_2)_* = (g_2)_* = v_2^{tp^n+1}$. The lemma is proved.

Now we begin to do the second induction. As an induction hypothesis, suppose that for $3 \leqslant k < p$, there exists $f_{k-1} \in [\Sigma^m V_{(k-1)p^n}; V_{(k-1)p^n}]$ such that $(f_{k-1})_* = v_2^{tp^n+1}$ and

$$h_{k-1} f_1 = f_{k-1} h_{k-1}, \qquad h'_{k-1} f_{k-1} = f_1 h'_{k-1} \tag{4.5}$$

$$\psi_{1,k-1} f_1 = f_{k-1}\psi_{1,k-1},\ \rho_{k-1,1} f_{k-1} = f_1 \rho_{k-1,1} \tag{4.6}$$

If $k = 3$, the above condition holds from Lemma 4.3 and (4.2).

Then there exist g_k, g'_k such that the following diagrams commute

$$\begin{tikzcd}
V_{p^n} \arrow[r, "h_{k-1}"] & \Sigma^{p^nq+1} V_{(k-1)p^n} \arrow[r, "\psi_{k-1,k}"] & \Sigma V_{kp^n} \arrow[r, "\rho_{k,1}"] & \Sigma V_{p^n} \\
\Sigma^m V_{p^n} \arrow[r, "h_{k-1}"] \arrow[u, "f_1"] & \Sigma^{m+p^nq+1} V_{(k-1)p^n} \arrow[r, "\psi_{k-1,k}"] \arrow[u, "f_{k-1}"] & \Sigma^{m+1} V_{kp^n} \arrow[r, "\rho_{k,1}"] \arrow[u, "g_k"] & \Sigma^{m+1} V_{p^n} \arrow[u, "f_1"]
\end{tikzcd} \tag{4.7}$$

$$\begin{tikzcd}
V_{(k-1)p^n} \arrow[r, "h'_{k-1}"] & \Sigma^{(k-1)p^nq+1} V_{p^n} \arrow[r, "\psi_{1,k}"] & \Sigma V_{kp^n} \arrow[r, "\rho_{k,k-1}"] & \Sigma V_{(k-1)p^n} \\
\Sigma^m V_{(k-1)p^n} \arrow[r, "h'_{k-1}"] \arrow[u, "f_{k-1}"] & \Sigma^{m+(k-1)p^nq+1} V_{p^n} \arrow[r, "\psi_{1,k}"] \arrow[u, "f_1"] & \Sigma^{m+1} V_{kp^n} \arrow[r, "\rho_{k,k-1}"] \arrow[u, "g'_k"] & \Sigma^{m+1} V_{(k-1)p^n} \arrow[u, "f_{k-1}"]
\end{tikzcd} \tag{4.8}$$

Since $(f_{k-1})_* = v_2^{tp^n+1}$, then $(g'_k)_* = v_2^{tp^n+1} + v_1^{(k-1)p^n}x$, it follows from Corollary 2.3(1) that $(g'_k)_* = v_2^{tp^n+1}$.

It follows from diagram (4.7) that $(g_k)_* = v_2^{tp^n+1} + v_1^{p^n}x$ for some $x \in BP_*$ and $v_1^{p^n}(g_k)_* = v_1^{p^n}v_2^{tp^n+1}$, then $v_1^{2p^n}x = 0 \mod(p, v_1^{kp^n})$, then it follows from Corollary 2.3(1) that $(g_k)_* = v_2^{tp^n+1}$.

Now it follows from (4.7) and (4.6) that $g_k\psi_{1,k} = g_k\psi_{k-1,k}\psi_{1,k-1} = \psi_{k-1,k}f_{k-1}\psi_{1,k-1} = \psi_{1,k}f_1$ and from (4.8), (4.6) that $\rho_{k,1}g'_k = \rho_{k-1,k}\rho_{k,k-1}g'_k = \rho_{k-1,1}f_{k-1}\rho_{k,k-1} = f_1\rho_{k,1}$, then there exist f'_{k-1} and f''_{k-1} such that the following diagrams commute

$$\begin{tikzcd}
V_{(k-1)p^n} \arrow[r, "h'_{k-1}"] & \Sigma^{(k-1)p^nq+1} V_{p^n} \arrow[r, "\psi_{1,k}"] & \Sigma V_{kp^n} \arrow[r, "\rho_{k,k-1}"] & \Sigma V_{(k-1)p^n} \\
\Sigma^m V_{(k-1)p^n} \arrow[r, "h'_{k-1}"] \arrow[u, "f'_{k-1}"] & \Sigma^{m+(k-1)p^nq+1} V_{p^n} \arrow[r, "\psi_{1,k}"] \arrow[u, "f_1"] & \Sigma^{m+1} V_{kp^n} \arrow[r, "\rho_{k,k-1}"] \arrow[u, "g_k"] & \Sigma^{m+1} V_{(k-1)p^n} \arrow[u, "f'_{k-1}"]
\end{tikzcd} \tag{4.9}$$

$$\begin{tikzcd}
V_{p^n} \arrow[r, "h_{k-1}"] & \Sigma^{p^nq+1} V_{(k-1)p^n} \arrow[r, "\psi_{k-1,k}"] & \Sigma V_{kp^n} \arrow[r, "\rho_{k,1}"] & \Sigma V_{p^n} \\
\Sigma^m V_{p^n} \arrow[u, "f_1"] \arrow[r, "h_{k-1}"] & \Sigma^{m+p^nq+1} V_{(k-1)p^n} \arrow[u, "f''_{k-1}"] \arrow[r, "\psi_{k-1,k}"] & \Sigma^{m+1} V_{kp^n} \arrow[u, "g'_k"] \arrow[r, "\rho_{k,1}"] & \Sigma^{m+1} V_{p^n} \arrow[u, "f_1"]
\end{tikzcd} \tag{4.10}$$

From (4.8) and (4.9), we have $h'_{k-1}(f'_{k-1} - f_{k-1}) = 0$ and $\rho_{k,k-1}(g_k - g'_k) = (f'_{k-1} - f_{k-1})\rho_{k,k-1}$, then $0 = \rho_{k-1}(g_k - g'_k) = \rho_{k-1,1}(f'_{k-1} - f_{k-1})\rho_{k,k-1}$. Then it follows from the following Lemma 4.13 that

$$(f'_{k-1} - f_{k-1})h_{k-1} = 0 \tag{4.11}$$

From (4.7), (4.10) we have $(f_{k-1} - f''_{k-1})h_{k-1} = 0$ and $\psi_{k-1,k}(f_{k-1} - f''_{k-1}) = (g_k - g'_k)\psi_{k-1,k}$, then $\psi_{k-1,k}(f_{k-1} - f''_{k-1})\psi_{1,k-1} = (g_k - g'_k)\psi_{1,k} = 0$, then it follows from Lemma 4.13 that

$$h'_{k-1}(f_{k-1} - f''_{k-1}) = 0. \tag{4.12}$$

(4.11) and (4.12) will be used later.

The proof of the following lemma will be postponed to the last of the proof of main theorem.

LEMMA 4.13: (1) *For* $g \in [\Sigma^t V_{(k-1)p^n}; V_{(k-1)p^n}]$ *such that* $\rho_{k-1,1} g i_{k-1} = 0$ *and* $h'_{k-1} g = 0$, *then* $g h_{k-1} = 0$.

(2) *For* g *as above such that* $\psi_{k-1,k} g \psi_{1,k-1} = 0$ *and* $g h_{k-1} = 0$, *then* $h'_{k-1} g = 0$.

Now we see that $(\bar{h}_k g_k - g'_k \bar{h}_k)\psi_{1,k} = i_k \pi_k g_k \psi_{1,k} - g'_k i_k \pi_k \psi_{1,k} = i_k \pi_k \psi_{1,k} f_1 - g'_k i_k \pi_1 = h_k f_1 - g'_k h_k$ and $\rho_{k,k-1}(h_k f_1 - g'_k h_k) = \rho_{k,k-1} i_k \pi_1 f_1 - \rho_{k,k-1} g'_k i_k \pi_1 = i_{k-1}\pi_1 f_1 - f_{k-1} i_{k-1} \pi_1 = 0$ (cf. inductive hypothesis (4.5)). Then similar to the proof of Lemma 4.3 there exists α such that $d'(\alpha) = 0$ and there exists β such that the following diagram commutes

$$\begin{tikzcd}
\Sigma^{kp^nq+1} V_{p^n} \arrow[r, "\psi_{1,k}"] & \Sigma^{p^nq+1} V_{kp^n} \arrow[r, "\rho_{k,k-1}"] & \Sigma^{p^nq+1} V_{(k-1)p^n} \arrow[r, "h'_{k-1}"] & \Sigma^{kp^nq+2} V_{p^n} \\
\Sigma^m V_{p^n} \arrow[u, "\alpha"] \arrow[ur, "h_k f_1 - g'_k h_k"] \arrow[r, "\psi_{1,k}"] & \Sigma^{m-(k-1)p^nq} V_{kp^n} \arrow[u, "\bar{h}_k g_k - g'_k \bar{h}_k"] \arrow[r, "\rho_{k,k-1}"] & \Sigma^{m-(k-1)p^nq} V_{(k-1)p^n} \arrow[u, "\beta"] \arrow[r] & \Sigma^{m+1} V_{p^n} \arrow[u, "x"]
\end{tikzcd} \tag{4.14}$$

On the other hand, $\rho_{k,1}(\bar{h}_k g_k - g'_k \bar{h}_k)\psi_{k-1,k} = \rho_{k,1} i_k \pi_k g_k \psi_{k-1,k} - \rho_{k,1} g'_k i_k \pi_k \psi_{k-1,k} = i_1 \pi_k \psi_{k-1,k} f_{k-1} - f_1 i_1 \pi_k \psi_{k-1,k} = h'_{k-1} f_{k-1} - f_1 h'_{k-1} = 0$, then there exists $\bar{x}$, $\bar{\beta}$ such that the following diagram commutes

$$\begin{tikzcd}
\Sigma^{kp^nq+1} V_{p^n} \arrow[r, "h_{k-1}"] & \Sigma^{(k+1)p^nq+2} V_{(k-1)p^n} \arrow[r, "\psi_{k-1,k}"] & \Sigma^{kp^nq+2} V_{kp^n} \arrow[r, "\rho_{k,1}"] & \Sigma^{kp^nq+2} V_{p^n} \\
\Sigma^m V_{p^n} \arrow[u, "\bar{\alpha}"] \arrow[r, "h_{k-1}"] & \Sigma^{m+p^nq+1} V_{(k-1)p^n} \arrow[u, "\bar{\beta}"'] \arrow[r, "\psi_{k-1,k}"] & \Sigma^{m+1} V_{kp^n} \arrow[u, "\bar{h}_k g_k - g'_k \bar{h}_k"] \arrow[r, "\rho_{k,1}"] & \Sigma^{m+1} V_{p^n} \arrow[u, "\bar{x}"]
\end{tikzcd} \tag{4.15}$$

From diagram (4.14), $\pi_1 \alpha = \pi_k \psi_{1,k} \alpha = \pi_k (h_k f_1 - g'_k h_k) = -\pi_k g'_k i_k \pi_1 = -\pi_k g'_k i_k \pi_{k-1} \psi_{1,k-1}$, then there exists η such that the following diagram commutes

$$\begin{tikzcd}
\Sigma^{kp^nq+1}V_{p^n} \arrow[r, "\pi_1"] & \Sigma^{(k+1)p^nq+2}V(0) \arrow[r, "\phi^{p^n}"] & \Sigma^{kp^nq+2}V(0) \arrow[r, "i_1"] & \Sigma^{kp^nq+2}V_{p^n} \\
\Sigma^m V_{p^n} \arrow[u, "-\alpha"] \arrow[r, "\psi_{1,k-1}"] & \Sigma^{m-(k-2)p^nq}V_{(k-1)p^n} \arrow[u, "\pi_k g'_k i_k \pi_{k-1}"] \arrow[r, "\rho_{k-1,k-2}"] & \Sigma^{m-(k-2)p^nq}V_{(k-2)p^n} \arrow[u, "\eta"] \arrow[r, "h'_{k-2}"] & \Sigma^{m+1}V_{p^n} \arrow[u, "-\alpha"]
\end{tikzcd} \tag{4.16}$$

Since

$$\begin{aligned}
\phi^{p^n}\pi_k g'_k i_k \pi_{k-1} &= \pi_{k-1}\rho_{k,k-1}g'_k i_k \pi_{k-1} = \pi_{k-1}f_{k-1}\rho_{k,k-1}i_k\pi_{k-1} \\
&= \pi_{k-1}f_{k-1}i_{k-1}\pi_{k-1} = \pi_k\psi_{k-1,k}f_{k-1}i_{k-1}\pi_{k-1} \\
&= \pi_k g_k \psi_{k-1,k}i_{k-1}\pi_{k-1} = \pi_k g_k i_k \phi^{p^n}\pi_{k-1} = \pi_k g_k i_k \pi_{k-2}\rho_{k-1,k-2},
\end{aligned}$$

then

$$\eta\rho_{k-1,k-2} = \pi_k g_k i_k \pi_{k-2}\rho_{k-1,k-2}$$

and there exists $\gamma \in [\Sigma^{m+1}V_{p^n}; \Sigma^{kp^nq+2}V(0)]$ such that

$$\eta = \pi_k g_k i_k \pi_{k-2} + \gamma i_1 \pi_{k-2}$$

and $-\alpha i_1 \pi_{k-2} = i_1\eta = i_1\pi_k g_k i_k \pi_{k-2} + i_1\gamma i_1\pi_{k-2}$, then we have $i_1\pi_k g_k i_k \pi_1 = -\alpha\bar{h}_1 - i_1\gamma\bar{h}_1$ and

$$i_1\pi_k g_k i_k \pi_1 = -\alpha\bar{h}_1 + \bar{h}_1 d'(i_1\gamma)\bar{h}_1 \tag{4.17}$$

since $0 = d'(\bar{h}_1 i_1\gamma) = i_1\gamma + \bar{h}_1 d'(i_1\gamma)$.

But $-\alpha\bar{h}_1 = \bar{h}_1\alpha = -i_1\pi_k g'_k i_k\pi_1$, then $i_1\pi_k(g_k + g'_k)i_k\pi_1 = i_1\pi_1 d'(i_1\gamma)i_1\pi_1$ and so

$$i_1\pi_k(g_k + g'_k - \psi_{1,k}d'(i_1\gamma)\rho_{k,1})i_k\pi_1 = 0 \tag{4.18}$$

Recall to (4.11), $\rho_{k,k-1}(g_k - g'_k)i_k\pi_1 = (f'_{k-1} - f_{k-1})\rho_{k,k-1}i_k\pi_1 = (f'_{k-1} - f_{k-1})h_{k-1} = 0$, then there exists ε such that $d'(\varepsilon) = 0$ and the following diagram commutes

$$\begin{tikzcd}
\Sigma^{(k-1)p^nq}V_{p^n} \arrow[r, "\psi_{1,k}"] & V_{kp^n} \arrow[r, "\rho_{k,k-1}"] & V_{(k-1)p^n} \\
 & \Sigma^m V_{kp^n} \arrow[u, "g_k - g'_k - \psi_{1,k}d'(i_1\gamma)\rho_{k,1}"] & \\
 & \Sigma^m V(0) \arrow[u, "i_k"] & \\
 & \Sigma^{m-p^nq-1}V_{p^n} \arrow[u, "\pi_1"] \arrow[uuul, "\varepsilon"] \arrow[uuur, "0"'] &
\end{tikzcd}$$

It is easily seen that $\pi_1\varepsilon i_1 = \pi_k\psi_{1,k}\varepsilon i_1 = 0$, then $\bar{h}_1\varepsilon\bar{h}_1 = 0$ and then $\bar{h}_1\varepsilon = -\varepsilon\bar{h}_1$. On the other hand

$$\begin{aligned}
-\varepsilon\bar{h}_1 = \bar{h}_1\varepsilon &= i_1\pi_k\psi_{1,k}\varepsilon = i_1\pi_k(g_k - g'_k - \psi_{1,k}d'(i_1\gamma)\rho_{k,1})i_k\pi_1 \\
&= 2i_1\pi_k(g_k - \psi_{1,k}d'(i_1\gamma)\rho_{k,1})i_k\pi_1 \quad \text{(summing with (4.18))} \\
&= 2i_1\pi_k g_k i_k \pi_1 - 2\bar{h}_1 d'(i_1\gamma)\bar{h}_1 \\
&= -2\alpha\bar{h}_1 \qquad \text{(from (4.17))}
\end{aligned}$$

Then by acting d' we have $\varepsilon = 2\alpha$ and

$$\begin{aligned}(g_k - g'_k - \psi_{1,k}d'(i_1\gamma)\rho_{k,1})h_k = \psi_{1,k}\varepsilon &= 2\psi_{1,k}\alpha \\ &= 2(h_k f_1 - g'_k h_k)\end{aligned}$$

Hence if we let

$$f_k = \tfrac{1}{2}(g_k + g'_k - \psi_{1,k}d'(i_1\gamma)\rho_{k,1}) \tag{4.19}$$

we have $f_k h_k = h_k f_1$.

Similarly, recall to (4.12), $i_1\pi_k(g_k - g'_k)\psi_{k-1,k} = i_1\pi_k\psi_{k-1,k}(f_{k-1} - f''_{k-1}) = h'_{k-1}(f_{k-1} - f''_{k-1}) = 0$, then there exists $\bar{\varepsilon}$ with $d'(\bar{\varepsilon}) = 0$ such that the following diagram commutes

$$\begin{tikzcd}
 & \Sigma^{kp^nq+1}V_{p^n} & \\
 & \Sigma^{kp^nq+1}V(0) \arrow[u, "i_1"] & \\
 & V_{kp^n} \arrow[u, "\pi_k"] & \\
\Sigma^{m+p^nq}V_{(k-1)p^n} \arrow[r, "\psi_{k-1,k}"] \arrow[uuur, "0"] & \Sigma^m V_{kp^n} \arrow[r, "\rho_{k,1}"] \arrow[u, "g_k - g'_k + \psi_{1,k}d'(i_1\gamma)\rho_{k,1}"] & \Sigma^m V_{p^n} \arrow[uuul, "\bar{\varepsilon}"']
\end{tikzcd}$$

i.e. $\bar{\varepsilon}\rho_{k,1} = i_1\pi_k(g_k - g'_k + \psi_{1,k}d'(i_1\gamma)\rho_{k,1})$ and

$$\begin{aligned}\bar{\varepsilon}\bar{h}_1 &= \bar{\varepsilon}\rho_{k,1}i_k\pi_1 = i_1\pi_k(g_k - g'_k + \psi_{1,k}d'(i_1\gamma)\rho_{k,1})i_k\pi_1 \\ &= 2i_1\pi_k g_k i_k\pi_1 \quad \text{(by summing with (4.18))}\end{aligned}$$

Since $\bar{\alpha}\bar{h}_1 = \bar{\alpha}i_1\pi_1 = \bar{\alpha}\rho_{k,1}i_k\pi_1 = \rho_{k,1}(\bar{h}_k g_k - g'_k\bar{h}_k)i_k\pi_1 = i_1\pi_k g_k i_k\pi_1$, then $\bar{\varepsilon}\bar{h}_1 = 2\bar{\alpha}\bar{h}_1$ and by acting d' we have

$$\bar{\varepsilon} = -d'(\bar{\varepsilon}\bar{h}_1) = -2d'(\bar{\alpha}\bar{h}_1) = 2\bar{\alpha} - 2d'(\bar{\alpha})\bar{h}_1$$

and so

$$\begin{aligned}h'_k(g_k - g'_k + \psi_{1,k}d'(i_1\gamma)\rho_{k,1}) = \bar{\varepsilon}\rho_{k,1} &= 2\bar{\alpha}\rho_{k,1} \\ &= 2\rho_{k,1}(\bar{h}_k g_k - g'_k\bar{h}_k) = 2(h'_k g_k - f_1 h'_k)\end{aligned}$$

Hence $f_1 h'_k = h'_k f_k$ (cf. (4.19)).

Since $k < p$, the induced BP_* homomorphism $(d'(i_1\gamma)i_0)_* \in \mathrm{Ext}_{BP_*BP}^{0,m-(k-1)p^nq}(BP_*, BP_*/(p, v_1^{p^n})) = 0$ (cf. Corollary 2.4), then $d'(i_1\gamma)_* = 0$ and $(f_k)_* = \frac{1}{2}[(g_k)_* + (g'_k)_*] = v_2^{tp^{n+1}}$. Recall to (4.19), it is easily seen that $f_k\psi_{1,k} = \psi_{1,k}f_1$, $\rho_{k,1}f_k = f_1\rho_{k,1}$, then the second induction is completed.

Hence we get a self map $f_{p-1}\colon \Sigma^m V_{(p-1)p^n} \to V_{(p-1)p^n}$ such that $(f_{p-1})_* = v_2^{tp^{n+1}}$ and $h_{p-1}f_1 = f_{p-1}h_{p-1}$, $h'_{p-1}f_{p-1} = f_1 h'_{p-1}$, then there exists $g'_p\colon \Sigma^m V_{p^{n+1}} \to V_{p^{n+1}}$ such that $\rho_{p,p-1}g'_p = f_{p-1}\rho_{p,p-1}$. Then $(g'_p)_* = v_2^{tp^{n+1}} + v_1^{(p-1)p^n}x$ for some $x \in BP_*$. It follows from Corollary 2.3(2) that

$$v_1^{(p-1)p^n}x = \sum_{j=1}^{[n+1/2]} \lambda_j v_1^{p^{n+1} - p^{n+1-2j}}\bar{c}_1(a_j p^{n+1-2j})$$

where $\lambda_j \in Z_p$ and $a_j \geqslant 2$.

The proof of the following lemma will be postponed to the last of the proof of main theorem.

LEMMA 4.22. *Suppose that for $a \geqslant 2$, $0 \leqslant j \leqslant s$, $v_2^{ap^j}$ can be realized in $[\Sigma^* V_pj; V_pj]$, then $\bar{c}_1(ap^s)$ can be realized in $[\Sigma^* V_{p^s}; V_{p^s}]$ and $v_1^{p^n - p^s}\bar{c}_1(ap^s)$ can be realized in $[\Sigma^* V_{p^n}; V_{p^n}]$.*

From the first inductive hypothesis, for $k \leqslant n$, $v_2^{tp^k}$ can be realized by ξ_k in $[\Sigma^* V_{p^k}; V_{p^k}]$, then it follows from Lemma 4.22 that for $1 \leqslant j \leqslant [n + 1/2]$, $v_1^{p^{n+1} - p^{n+1-2j}}\bar{c}_1(a_j p^{n+1-2j})$ can be realized in $[\Sigma^* V_{p^{n+1}}; V_{p^{n+1}}]$, then there exists $\bar{g} \in [\Sigma^m V_{p^{n+1}}; V_{p^{n+1}}]$ such that $(\bar{g})_* = v_1^{(p-1)p^n} x$ and then $(\xi_{n+1})_* = v_2^{tp^{n+1}}$ if we put $\xi_{n+1} = g'_p - \bar{g}$. Then the first induction is also completed.

Now if $j \leqslant p^n$, let ρ: $V_{p^n} \to V_j$ be projection, then $\rho\xi_n i_0$: $\Sigma^{tp^n(p+1)q} S^o \to V_j$ induces BP_* homomorphism $(\rho\xi_n i_0)_* = v_2^{tp^n}$. Since $\rho\xi_n i_0$ can be extended to η_j: $\Sigma^{tp^n(p+1)q} V_j \to V_j$ such that $(\eta_j)_* = v_2^{tp^n}$, then it follows from Proposition 2.5 that $\beta_{tp^n/j} = \delta\delta_j v_2^{tp^n}$ survives to the following nontrivial map

$$\Sigma^{tp^n(p+1)q} S^o \xrightarrow{i_0} \Sigma^{tp^n(p+1)q} V_j \xrightarrow{\eta_j} V_j \xrightarrow{\pi} \Sigma^{jq+1} V(0) \xrightarrow{\tau} \Sigma^{jq+2} S^o$$

the main Theorem I is then proved.

In the following, we give the proof of Lemma 4.13 and 4.22.

Proof of Lemma 4.13. (1) Since $h'_{k-1} g = i_1 \pi_{k-1} g = 0$, then $i_1 \pi_{k-1} g i_{k-1} \pi_1 = 0$. It follows from Proposition. 3.20 that $\pi_{k-1} g i_{k-1} \pi_1 = 0$. Then there exists $u \in [\Sigma^t V_{p^n}; \Sigma^{p^n q+1} V(0)]$ such that $g i_{k-1}\pi_1 = i_{k-1} u$, and from Proposition 3.22, there exists $\bar{u} \in [\Sigma^t V_{p^n}; V_{p^n}]$ such that $d'(\bar{u}) = 0$ and $u = \pi_1 \bar{u}$ and so

$$\bar{h}_1 \bar{u} = i_1 \pi_1 \bar{u} = \rho_{k-1,1} i_{k-1} \pi_1 \bar{u} = \rho_{k-1,1} g i_{k-1} \pi_1 = 0$$

Then $\bar{u} = d'(\bar{h}_1 \bar{u}) = 0$ and so $g i_{k-1} \pi_1 = 0$.

The proof of the second half is similar.

Proof of Lemma 4.22. From Proposition 2.1, we have

$$\bar{c}_1(ap^s) = v_2^{ap^s} - a v_1^{b_1} v_2^{a_1 p^{s-2}} - 2a \sum_{j=2}^{s-2} v_1^{b_j} v_2^{a_j p^{s-1-j}} \bmod (p, v_1^{p^s})$$

where $b_j = p^s + p^{s-1} - p^{s-j} - p^{s-1-j}$ and $a_j = (ap-1)p^j + 1$, $j \geqslant 1$. From the hypothesis, $v_2^{ap^s}$ can be realized in $[\Sigma^* V_{p^s}; V_{p^s}]$ and $v_2^{a_j p^{s-1-j}}$ can be realized by a self map ε_j of $V_{p^{s-1-j}}$. Consider the cofibration

$$\Sigma^{b_j q} V_{p^{s-1-j}} \xrightarrow{\bar{\psi}} V_{p^s + p^{s-1} - p^{s-j}} \xrightarrow{\bar{\rho}} V_{b_j} \xrightarrow{h} \Sigma^{b_j q+1} V_{p^{s-1-j}}$$

in which $(\bar{\psi})_* = v_1^{b_j}$ and let ρ': $V_{p^s + p^{s-1} - p^{s-j}} \to V_{p^s}$, then $(\rho'\bar{\psi}\varepsilon_j i_0)_* = v_1^{b_j} v_2^{a_j p^{s-1-j}} \in \mathrm{Ext}^0(BP_*, BP_*/(p, v_1^{p^s}))$. Since $\rho'\bar{\psi}\varepsilon_j i_0$ can be extended to a self map of V_{p^s} which has the same BP_* homomorphism, then $v_1^{b_j} v_2^{a_j p^{s-1-j}}$ can be realized by a self map of V_{p^s}, then $\bar{c}_1(ap^s)$ can be realized by a self map of V_{p^s}. By a similar method we can prove that $v_1^{p^n - p^s}\bar{c}_1(ap^s)$ can be realized in $[\Sigma^* V_{p^n}; V_{p^n}]$. The lemma is then proved.

Proof of Theorem II. Recall that we have a cofibration

$$S^o \xrightarrow{p} S^o \xrightarrow{j_0} V(0) \xrightarrow{\tau} \Sigma S^o$$

and $\delta_{V(0)} = j_0 \tau$: $V(0) \to \Sigma V(0)$ (we will write $\delta_{V(0)}$ as δ if there is no confusion). Let $C(\delta)$ be the cofibre of δ, then it follows from 3×3 lemma that there is a commutative diagram as follows

$$\begin{tikzcd}[column sep=small]
\Sigma^{-1}V(0) \arrow[rr, "\delta"] \arrow[dr, "\tau"] & & V(0) \arrow[rr, "\tau"] \arrow[dr, "\lambda"] & & \Sigma S^o \arrow[rr, "p^2"] \arrow[dr, "p"] & & \Sigma S^o \\
& S^o \arrow[ur, "j_0"] \arrow[dr, "p"] & & C(\delta) \arrow[ur, "\tau'"] \arrow[dr, "\rho"] & & S^o \arrow[ur, "p"] \arrow[dr, "j_0"] & \\
S^o \arrow[ur, "p"] \arrow[rr, "p^2"] & & S^o \arrow[rr, "j_0"] \arrow[ur, "j'"] & & V(0) \arrow[rr, "\delta"] \arrow[ur, "\tau"] & & \Sigma V(0)
\end{tikzcd}$$

and we find that $c(\delta) = C(p^2)$ which we write as $M(p^2)$, i.e. there are cofibrations

$$S^o \xrightarrow{p^2} S^o \xrightarrow{j'} M(p^2) \xrightarrow{\tau'} \Sigma S^o$$

$$\Sigma^{-1} V(0) \xrightarrow{\delta} V(0) \xrightarrow{\lambda} M(p^2) \xrightarrow{\rho} V(0)$$

which realize the following short exact sequences

$$0 \longrightarrow BP_* \xrightarrow{p^2} BP_* \xrightarrow{j'_*} BP_*/(p^2) \to 0$$

$$0 \longrightarrow BP_*/(p) \xrightarrow{\lambda_* = p} BP_*/(p^2) \xrightarrow{\rho_*} BP_*/(p) \longrightarrow 0$$

and we have $\tau = \tau'\lambda$, $j_0 = j'\rho$.

From (3.5) we have $\delta_{V_j}\colon V_j \to \Sigma V_j$ such that $\delta_{V_j} i = i\delta_{V(0)}$ then there exists i' such that two bottom rows of the following diagram is a commutative diagram of cofibrations

$$\begin{tikzcd}
& \Sigma^{jq+1} V(0) \arrow[r, "\lambda'' = \lambda"] & \Sigma^{jq+1} W = \Sigma^{jq+1} M(p^2) & \\
\Sigma^{-1} V_j \arrow[r, "\delta"] & V_j \arrow[r, "\lambda'"] \arrow[u, "\pi"] & C(\delta_{V_j}) \arrow[r, "\rho'"] \arrow[u, "\pi'"] & V_j \\
\Sigma^{-1} V(0) \arrow[r, "\delta_{V(0)}"] \arrow[u, "i"] & V(0) \arrow[r, "\lambda"] \arrow[u, "i"] & M(p^2) \arrow[r, "\rho"] \arrow[u, "i'"] & V(0) \arrow[u, "i"]
\end{tikzcd}$$

Since $\rho_*\colon BP_* M(p^2) \to BP_* V(0)$, $i_*\colon BP_* V(0) \to BP_* V_j$ are all canonical projection, then i'_*, ρ'_* also are. Then from $\lambda_* = p$, we get $\lambda'_* = p$ and the middle row realizes the following short exact sequence (write $C(\delta_{V_j}) = M(p^2, v_1^j)$)

$$0 \to BP_*/(p, v_1^j) \xrightarrow{p} BP_*/(p^2, v_1^j) \to BP_*/(p, v_1^j) \to 0$$

Since i' realizes the canonical projection $BP_*/(p^2) \to BP_*/(p^2, v_1^j)$, then if W is the cofibre of i', then $BP_* W = BP_*/(p^2)$ and it follows from the uniqueness that $W = M(p^2)$ and there is λ'' which is compatible with λ such that $\lambda''\pi = \pi'\lambda'$ or $\lambda\pi = \pi'\lambda'$.

If $j \leqslant p^{n-1}$ and $j \equiv 0 \pmod p$, then $j = kp^s (s \geqslant 1)$ and it follows from corollary 3.13 that V_j is a split ring spectrum over Z_p. In the proof of Theorem I, we have had a self map

$$\eta_j\colon \Sigma^{tp^{n-1}(p+1)q} V_j \to V_j$$

such that $(\eta_j)_* = v_2^{tp^{n-1}}$ and we may assume $\eta_j \in \mathscr{C}_*(V_j)$ since its components in $\delta\mathscr{C}_*$, $\bar{h}\mathscr{C}_*$, $\bar{h}\delta\mathscr{C}_*$ will induce zero homomorphism. Then it follows from (3.19) that $\delta\eta_j^p = \eta_j^p\delta$ and then there exists η' such that the following diagram commutes

$$\begin{tikzcd}
\Sigma^{-1}V_j \arrow[r, "\delta"] & V_j \arrow[r, "\lambda'"] & M(p^2, v_1^j) = C(\delta_{V_j}) \\
\Sigma^{tp^n(p+1)q}V_j \arrow[r, "\delta"] \arrow[u, "\eta_j^p"] & \Sigma^{tp^n(p+1)q}V_j \arrow[r, "\lambda'"] \arrow[u, "\eta_j^p"] & \Sigma^{tp^n(p+1)q}M(p^2, v_1^j) \arrow[u, "\eta'"]
\end{tikzcd}$$

Then

$$\beta_{tp^n/j} = \tau\pi\eta_j^p ij_0 = \tau'\lambda\pi\eta_j^p ij_0 = \tau'\pi'\lambda'\eta_j^p ij_0 = \tau'\pi'\eta'\lambda' ij_0$$
$$= \tau'\pi'\eta' i'\lambda j_0 = \tau'\pi'\eta' i' j' p,$$

where we use the equation $\tau = \tau'\lambda$, $\lambda j_0 = j'p$ which are easily seen from the following commutative diagram

$$\begin{tikzcd}
S^o \arrow[r, "p"] \arrow[d, equal] & S^o \arrow[r, "j_0"] \arrow[d, "p"] & V(0) \arrow[r, "\tau"] \arrow[d, "\lambda"] & \Sigma S^o \\
S^o \arrow[r, "p^2"] & S^o \arrow[r, "j'"] & M(p^2) \arrow[ur, "\tau'"'] &
\end{tikzcd}$$

So $\beta_{tp^n/j}$ are divisible by p. Theorem II is then proved.

Theorem III can be obtained from Theorem I directly.

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Department of Mathematics
Nankai University
Tianjin
People's Republic of China